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The relaxation strength and the relaxation time of anelasticity experiments on foils or thin plates

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Abstract

The relaxation strength measured in bending experiments on anelastic foils can substantially differ from the standard result that is applicable in the case of long rods. We analyze the relaxation strength of foil-shaped samples, presenting quantitative results for the two limiting situations of a long and a short foil. For cubic crystals, we discuss the relaxation strengths of two well-known relaxation processes: the Snoek effect and the Gorsky effect. We consider the influence of the foil geometry on the diffusion coefficient (or relaxation time) obtained in Gorsky effect measurements. Our results for the relaxation strength and the diffusion coefficient in Gorsky effect experiments are a generalization of previous calculations which apply to only two special crystal orientations.

1. Introduction

Studies of anelasticity represent an important technique for the investigation of structural relaxation processes in solids [1, 2]. A chief experimental result is the relaxation strength which provides insight into the microscopic nature of the relaxation process, caused, for instance, by reorientation or diffusion of lattice defects. The relaxation time, however, yields information on the dynamics of the investigated relaxation process.

Figure 1 shows a frequent experimental situation, the bending of a foil or thin plate (l_x, $l_y > l_z$). The foil is clamped at one end in cantilever fashion, and its relaxation behavior is investigated either quasi-statically from its time-dependent bending (elastic after-effect) or from the frequency and the damping of its flexural vibrations (vibrating-reed technique). The bending or the vibrations are induced by a bending moment around the *Y* direction caused, for instance, by a force acting in the *Z* direction on the free end of the foil. Examples of such studies can be found in refs. 3–14.

The anelastic relaxation of the sample in Fig. 1 is determined by how its flexural rigidity depends on its elastic compliances. This dependence, and its variation with crystal orientation of the sample, is standard knowledge for a long rod [1, 2]. For a foil, however, the flexural rigidity can depend in a different way on the elastic compliances. Bausch *et al.* [15] and Wagner [16] discussed the effects of the foil geometry for the Gorsky relaxation [17, 18], and their theoretical predictions were experimentally confirmed by Tretkowski *et al.* [4]. However, the theoretical results of Bausch

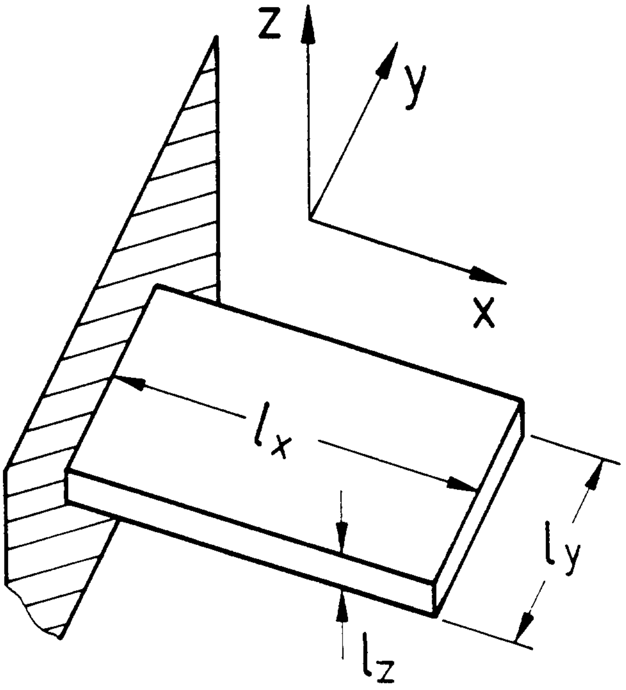

Fig. 1. Geometrical situation for an anelastic bending experiment on a foil. The bending moment on the sample acts in the *Y* direction, caused, for instance, by a force in the *Z* direction on the free end of the sample.

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et al. [15] and of Wagner [16] are restricted to two crystal orientations of the sample so that they are not generally applicable.

In this paper, we derive expressions for the relaxation strength in a bending experiment on foils of different geometry, applicable to arbitrary crystal orientations. For the special case of cubic crystals, we discuss further the relaxation strength of two well-known relaxation processes: the Snoek effect and the Gorsky effect [1, 2, 17]. For the Gorsky effect, we consider finally also the influence of the foil geometry on the measured relaxation time. A more detailed discussion of possible sample geometries and deformation models will be given elsewhere [19].

2. Determination of the relaxation strength

In a bent foil with cartesian coordinates as shown in Fig. 1, the stresses σ_{zz}, σ_{yz} and σ_{xz} can be considered to be zero [20]. The relation $\sigma_{xy}=0$ holds additionally since no torque is applied around the X direction. A decisive criterion for the bending behavior of the foil is the influence of the clamp. In regions of the foil that are close to the clamp, the clamp will not allow any bending around the X axis, such as is expected in the absence of the clamp as a consequence of lateral contractions and dilatations in the Y direction. We can distinguish two limiting cases: the case of a long foil, $l_x \gg l_y$, and the case of a short foil, $l_x \ll l_y$. For a short foil, the clamp suppresses any significant bending around the X axis within the entire foil, so that we can assume that no strain ϵ_{yy} exists. This limiting situation was considered in the calculations of Bausch *et al.* [15] and Wagner [16]. In the experiments of Tretkowski *et al.* [4], which confirmed these calculations, a bending of the foil around the X axis was particularly effectively suppressed since clamps were mounted at both ends of the foil. For a long foil, on the contrary, we can neglect the regions of the foil which are influenced by the clamp so that we can assume that $\sigma_{yy}=0$. In the following, we discuss the two limiting cases above, pointing out at the same time that the general relaxation behavior will – depending on the ratio of l_x to l_y – exhibit a continuous transition between the two limiting cases.

We consider first the bending of a long foil ($l_x \gg l_y \gg l_z$). The bending moment in the Y direction is caused, for instance, by a force acting in the Z direction on the free sample end. The bending moment and the resulting bending of the foil are proportional to the derivative of the stress σ_{xx} and the strain ϵ_{xx} with respect to the Z coordinate. Since $\sigma_{yy}=\sigma_{zz}=\sigma_{yz}=\sigma_{xz}=\sigma_{xy}=0$ holds for a long foil, the bending behavior is completely determined by the relation

$$\epsilon_{xx}=s_{11}\sigma_{xx}=E^{-1}\sigma_{xx} \tag{1}$$

where the elastic compliance s_{11} (in Voigt's notation [21]) is defined in the cartesian coordinates in Fig. 1, and where $E^{-1}(=s_{11})$ is the reciprocal Young's modulus in the X direction. The relaxation strength Δ_E is determined by the relaxation $\delta s_{11}=\delta(E^{-1})$ of $s_{11}=E^{-1}$, so that it is given by

$$\Delta_E=\frac{\delta s_{11}}{s_{11}}=\frac{\delta(E^{-1})}{E^{-1}} \tag{2}$$

Equation (2) shows that the relaxation strength of the long foil is identical with the standard result for a long rod [1, 2].

Let us now consider the bending of a short foil ($l_x \ll l_y$). To calculate the relaxation strength in this case, we need again a relationship between ϵ_{xx} and σ_{xx}, similarly as in eqn. (1). Considering (i) that the only non-zero stresses are σ_{xx} and σ_{yy} and (ii) that the strain ϵ_{yy} is zero, we can write

$$\epsilon_{xx}=s_{11}\sigma_{xx}+s_{12}\sigma_{yy} \qquad \epsilon_{yy}=0=s_{21}\sigma_{xx}+s_{22}\sigma_{yy} \tag{3}$$

Equation (3) describes a situation between a uniaxial stress and a uniaxial strain. According to this equation, the relation between ϵ_{xx} and σ_{xx} reads (note that $s_{12}=s_{21}$)

$$\epsilon_{xx}=\left(s_{11}-\frac{s_{12}{}^2}{s_{22}}\right)\sigma_{xx}=E_{\text{short}}{}^{-1}\sigma_{xx} \tag{4}$$

where the quantity $E_{\text{short}}{}^{-1}=s_{11}-s_{12}{}^2/s_{22}$ is an effective reciprocal Young's modulus in the X direction, valid for the bending of a short foil. Equation (4) shows that the bending behavior of a short foil differs from that of a long foil (and a long rod), and that $E_{\text{short}}{}^{-1}$ is smaller than $s_{11}=E^{-1}$ (s_{22} is positive for stability reasons [22]).

The relaxation strength $\Delta_{E_{\text{short}}}$ observed in the case of the short foil is determined by the relaxations δs_{11}, δs_{12} and δs_{22} of the compliances contributing to $\Delta_{E_{\text{short}}}$. Accordingly, the relaxation strength $\Delta_{E_{\text{short}}}$ of the short foil is given by

$$\Delta_{E_{\text{short}}}=\frac{E_{\text{short}}{}^{-1}(s_{ij}+\delta s_{ij})-E_{\text{short}}{}^{-1}(s_{ij})}{E_{\text{short}}{}^{-1}(s_{ij})} \tag{5}$$

3. Cubic crystal symmetry

The anelastic properties of cubic crystals are completely determined by the three compliances S_{11} ($=S_{22}=S_{33}$), S_{12} ($=S_{13}=S_{23}$) and S_{44} ($=S_{55}=S_{66}$) and by their anelastic relaxations δS_{11}, δS_{12} and δS_{44} [1, 2, 20–22]. These compliances are defined in the coordinate system of the cubic crystal axes, and they will be written in capital letters to discriminate them from the compliances in the cartesian coordinates in Fig. 1 (lower-case letters). An alternative description is possible with the help of the bulk modulus $B = 1/3(S_{11} + 2S_{12})$ and the two shear compliances $S' = 2(S_{11} - S_{12})$ and S_{44}, again together with their respective anelastic relaxations. The two shear compliances are identical for elastic isotropy. In the following, we shall also use an anisotropy compliance $S_A = S' - S_{44} = 2S_{11} - 2S_{12} - S_{44}$.

The compliances s_{ij} can be expressed by the compliances S_{kl} with the help of the cosines a_{mn} between the X, Y and Z axes of the respective coordinate systems (the first and the second subscript $m = 1$, 2 or 3 and $n = 1$, 2 or 3 denote the coordinate axes X, Y or Z in Fig. 1 and the crystal axes X, Y or Z respectively). Note also the relation $a_{k1}^2 + a_{k2}^2 + a_{k3}^2 = a_{1k}^2 + a_{2k}^2 + a_{3k}^2 = 1$ [21] which is used in some of the subsequent calculations.

In tensor notation, the elastic compliances s_{ijkl} can be expressed by S_{11}, S_{12} and S_A according to

$$\begin{aligned} s_{ijkl} = {} & S_{12}\delta_{ij}\delta_{kl} + \tfrac{1}{4}S_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ & + \tfrac{1}{2}S_A \sum_{\mu} a_{i\mu}a_{j\mu}a_{k\mu}a_{l\mu} \end{aligned} \tag{6}$$

where δ_{mn} is 1 for $m = n$ and is 0 otherwise. Equation (6) can be used to calculate the s_{ijkl} and δs_{ijkl} from S_{kl} and δS_{kl}, which in turn allows calculation of the relaxation strengths in eqns. (2) or (5).

Polycrystalline samples are macroscopically isotropic so that their elastic properties are completely determined by only two moduli, for instance the bulk modulus B and the shear modulus G. For such samples, s_{11}, s_{12} and s_{22} can be written as [1, 2, 20, 22]

$$s_{11} = s_{22} = \frac{1}{E} = \frac{1}{9B} + \frac{1}{3G} \qquad s_{12} = \frac{1}{9B} - \frac{1}{6G} \tag{7}$$

where the bulk modulus is $B = 1/3(S_{11} + 2S_{12})$. To establish the dependence of G on S_{11}, S_{12} and S_{44} involves a complex averaging procedure between $S' = 2(S_{11} - S_{12})$ and S_{44} [1, 2]. The actual value lies between the orientational average over the shear moduli, yielding $G = 2/5S' + 3/5S_{44}$ (Voigt's average), and the orientational average over the shear compliances, yielding $G^{-1} = \frac{2}{5}S' + \frac{3}{5}S_{44}$ (Reuss'average) [1, 2, 22]. For both limits, it is possible to determine the relaxation strength from the compliances S_{kl} and their relaxations δS_{kl}.

4. Snoek effect

The Snoek effect is an anelastic relaxation caused by the reorientation of defects having a lower point symmetry than the crystal, for instance by octahedral or tetrahedral interstitial atoms in b.c.c. metals [1, 2, 17]. In the latter case, and for small defect concentrations, the relaxations δS_{ij} of the compliances S_{ij} are given by [17]

$$\delta S_{11} = -2\,\delta S_{12} = \frac{2\rho(S_{11} - S_{12})^2(A - B)^2}{9k_B T} \qquad \delta S_{44} = 0 \tag{8}$$

where ρ is the number of defects per volume, $k_B T$ is the thermal energy and $A - B$ is the difference of the trace components of the double-force tensor of the defects.

The size of the relaxation process is completely determined by δS_{11}, so that this quantity will be used for the description of this Snoek relaxation. With the help of δS_{11} and eqns. (2), (4), (5) and (8), the relaxation strength of a long foil is given by

$$\Delta_E = \frac{\delta s_{11}}{s_{11}} = \frac{1 - 3\Gamma_{11}}{S_{11} - S_A\Gamma_{11}}\,\delta S_{11} \tag{9}$$

whereas for a short foil it is

$$\Delta_{E_\text{short}} = \frac{s_{22}^2(1 - 3\Gamma_{11}) + s_{12}s_{22}(1 - 3\Gamma_{12}) + s_{12}^2(1 - 3\Gamma_{22})}{s_{22}(s_{11}s_{22} - s_{12}^2)}\,\delta S_{11} \tag{10}$$

In these equations, the quantities Γ_{11}, Γ_{12} and Γ_{22} are given by

$$\Gamma_{11} = a_{11}^2 a_{12}^2 + a_{11}^2 a_{13}^2 + a_{12}^2 a_{13}^2 \tag{11}$$

$$\Gamma_{12} = a_{11}^2 a_{21}^2 + a_{12}^2 a_{22}^2 + a_{13}^2 a_{23}^2 \tag{12}$$

and

$$\Gamma_{22} = a_{21}^2 a_{22}^2 + a_{21}^2 a_{23}^2 + a_{22}^2 a_{23}^2 \tag{13}$$

The relaxation strength for the long foil is identical with the standard result for a long rod [1, 2]. Further, we point out that the expression for Δ_{E_short} follows from a linear expansion of E_short^{-1} with respect to δs_{ij} (or S_{11}), so that it is valid only for small δS_{11}. The expression for Δ_E, on the contrary, does not involve such an expansion so that it describes also large relaxations δS_{11}.

5. Gorsky effect

The Gorsky effect is an anelastic relaxation resulting from long-range diffusion of crystal defects which expand or contract the lattice [1, 15–18]. For defects with an isotropic double-force tensor $P\delta_{ij}$, the Gorsky relaxation of the elastic compliances S_{11}, S_{12} and S_{44} can be written as [17]

$$\delta S_{11}=\delta S_{12}=\frac{P^2}{9B^2\delta\mu/\delta\rho} \qquad \delta S_{44}=0 \tag{14}$$

where $\delta\mu/\delta\rho$ is the derivative of the chemical potential of the defects with respect to the number ρ of defects per unit volume. An isotropic double-force tensor is well justified for hydrogen interstitials in f.c.c. and b.c.c. metals [18, 23], and it was also assumed in the previous theoretical treatments of the Gorsky relaxation [15–17].

According to eqns. (2), (4), (5) and (14), the relaxation strength of a long foil is given by

$$\Delta_E=\frac{\delta s_{11}}{s_{11}}=\frac{P^2}{9s_{11}B^2\ \delta\mu/\delta\rho} \tag{15}$$

whereas in the case of a short foil it is

$$\Delta_{E_{\mathrm{short}}}=\frac{(s_{22}-s_{12})^2}{s_{11}s_{22}-s_{12}^2}\ \frac{P^2/9s_{22}B^2}{\delta\mu/\delta\rho+P^2/9s_{22}B^2} \tag{16}$$

Equation (15) agrees again with the standard result for a long rod. We mention specifically that the expressions in both eqn. (15) and eqn. (16) do not involve any expansion so that they are valid for arbitrarily large relaxations. The two relaxation strengths differ in that Δ_E is inversely proportional to $\delta\mu/\delta\rho$ whereas $\Delta_{E_{\mathrm{short}}}$ depends on this quantity in a more complicated way. However, in the low concentration limit $\rho\rightarrow 0$, $\delta\mu/\delta\rho$ assumes usually the diverging value k_BT/ρ so that both relaxation strengths become inversely proportional to k_BT.

The relaxation strength $\Delta_{E_{\mathrm{short}}}$ was calculated by Bausch *et al.* [15] and Wagner [16] for two special crystal orientations. Their result was written as

$$\Delta_{E_{\mathrm{short}}}=\frac{R\omega}{a-\omega}=\frac{R\omega}{\delta\mu/\delta\rho+P^2/B-\omega} \tag{17}$$

where the second term on the right-hand side of eqn. (14) is valid since $a=\delta\mu/\delta\rho+P^2/B$ (see also refs. 24 and 25). The comparison of eqns. (16) and (17) shows that the two quantities ω and R can be given in the form

$$\omega=\frac{P^2}{B}\left(1-\frac{1}{9s_{22}B}\right) \qquad R=\frac{(s_{22}-s_{12})^2}{s_{11}s_{22}-s_{12}^2}\ \frac{1}{9s_{22}B-1} \tag{18}$$

which holds for arbitrary crystal directions.

The relaxation time in Gorsky effect experiments depends on the chemical diffusion coefficient D of the defects. It was first pointed out by Bausch *et al.* [15] and Wagner [16] that the diffusion coefficient D_{short} obtained from experiments on short foils differs from D if D_{short} is simply defined according to Fick's law. The fact that D_{short} differs from D results from inhomogeneous coherency stresses within the thin foil. Such coherency stresses, which arise in the neighborhood of the clamp, become unimportant in the case of a long foil, so that the diffusion coefficient which is measured in experiments on long foils is identical with the chemical diffusion coefficient D. Bausch *et al.* [15] and Wagner [16] calculated the relation between D_{short} and D for two special crystal orientations of the sample. In the following, we give an expression for D_{short} that applies also to arbitrary crystal orientations. This expression is presented without explicit derivation since this will be done in detail in a later publication [19].

The general relation between D_{short} and D can be written as

$$\frac{D_{\mathrm{short}}}{D}=1+\frac{P^2}{9s_{22}B^2\ \delta\mu/\delta\rho}=1+\frac{P^2/B-\omega}{\delta\mu/\delta\rho} \tag{19}$$

where ω is defined according to eqn. (18). Equation (19) shows that D_{short} is always larger than D since s_{22} is positive for stability reasons. A very important fact is finally, that D_{short} and D become identical in the limit of small defect concentrations, *i.e.* for $\rho\rightarrow 0$, where $\delta\mu/\delta\rho$ assumes the diverging value k_BT/ρ. This means that the influence of coherency stresses becomes negligible in the low concentration limit.

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